

ON THE GEOGRAPHY OF THREEFOLDS OF GENERAL TYPE

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ABSTRACT. Let X be a complex nonsingular projective 3-fold of general type. We show that there are positive constants c , c' and m_1 such that $\chi(\omega_X) \geq -c\text{Vol}(X)$ and $P_m(X) \geq c'm^3\text{Vol}(X)$ for all $m \geq m_1$.

1. INTRODUCTION AND KNOWN RESULTS

The birational classification of surfaces of general type is well understood. For example, it is known that if X is a surface of general type then $|mK_X|$ induces a birational map for all $m \geq 5$. As a general rule, it is not possible to classify surfaces of general type with given invariants. In general, the best that one can do is to show that the invariants of a surface X satisfy certain inequalities. A fundamental inequality for the invariants of a minimal surface of general type is the Bogomolov-Miyaoka-Yau inequality $K_X^2 \leq 9\chi(\mathcal{O}_X)$.

It is a natural problem to try and extend the results for surfaces to higher dimensions. There have been many partial results for 3-folds. For example, it is shown in [6] that if X is a Gorenstein minimal 3-fold of general type, then $|mK_X|$ induces a birational map for all $m \geq 5$. In fact the proof is based upon the fact that for such 3-folds, we have the Miyaoka-Yau inequality $K_X^3 \leq 72\chi(\omega_X)$.

Despite many partial results, the geometry of non-Gorenstein 3-folds of general type has proven to be a very challenging topic. In a recent paper however, the first author and M. Chen [2] show the remarkable result that if X is a smooth complex projective 3-fold of general type, then $P_{12} \geq 1$, $P_{24} \geq 2$ and $|mK_X|$ induces a birational map for all $m \geq 77$. It is then natural to hope that further precise results on the geography of 3-folds of general type may be within reach. The purpose of this paper is to show that using the methods of [2] one can in fact prove an inequality similar to the Miyaoka-Yau inequality which holds for non-Gorenstein 3-folds of general type. Namely we show that

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Theorem 1. *There exists a constant $c > 0$ such that for any minimal 3-fold of general type with terminal singularities X , we have*

$$\chi(\omega_X) \geq -cK_X^3.$$

Recall that for any minimal 3-fold of general type with terminal singularities, we have $\text{Vol}(X) = K_X^3$. It should be noted that $\chi(\omega_X)$ may be negative for 3-folds of general type. In fact consider curves C_1 , C_2 and C_3 with genus g_i and involutions σ_i such that $C_i / \langle \sigma_i \rangle \cong \mathbb{P}^1$. Then the 3-fold X given by a desingularization of the quotient of $C_1 \times C_2 \times C_3$ by the “diagonal” involution, has $\chi(\omega_X) < 0$. In fact if we let $g_1 = g_2 = g$, then for fixed g_3 and for $g \gg 0$ one has that $-\chi(\omega_X) = O(g^2)$ and $K_X^3 = O(g^2)$. So the inequality of Theorem 1 has the right shape. The constant c that may be computed with the methods of this paper and the results of [2] is $c = 32 \cdot 120^3$. We expect that this is far from optimal and so we make no effort to determine it explicitly. We remark that if $\text{Vol}(X) \gg 0$, then using the results of [12], one can recover $c = 2502$.

We also prove the following result concerning the plurigenera of X .

Theorem 2. *There exist constants $c' > 0$ and $m_1 > 0$ such that for any minimal 3-fold of general type with terminal singularities X , we have*

$$P_m(X) \geq c'm^3K_X^3 \quad \text{for all } m \geq m_1.$$

Once again the values of c' and m_1 that may be computed with the methods of this paper are far from optimal so we make no effort to determine their values (note that using the results of [2], it follows from the arguments below that $c' = \frac{5}{89168}$ and $m_1 = 112$ suffice).

It would be interesting to:

- (1) Determine the optimal value of m_1 in Theorem 2;
- (2) See if Theorem 2 can be recovered by using the methods of [12];
- (3) See if Theorems 1 and 2 hold in higher dimensions.

We remark that the proof of the results of this paper is based on the methods of [2]. We have chosen to keep the exposition of this paper as self contained and simple as possible. Therefore, we include a proof of all the results of [2] that we will use (namely inequalities (1) and (2)).

2. SOME INEQUALITIES

In this section, we will prove the following inequalities:

Theorem 3. *Let X be a minimal 3-fold of general type with terminal singularities, then*

$$P_4 + P_5 + P_6 - 3P_2 - P_3 - P_7 \geq 0. \tag{1}$$

and

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi(\mathcal{O}_X) + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13} + 14\sigma_{12} \tag{2}$$

where σ_{12} is a positive integer that will be defined below.

Note that a stronger version of the above inequalities is proved in [2]. Here we include a simpler and self contained proof of this (weaker) version of the inequalities of [2]. The stronger version also follows from the methods of this paper, however it is not necessary for our purposes so we have chosen not to include it here.

We consider now X a minimal 3-fold of general type with terminal singularities. According to Reid (see last section of [11]), there is a “basket” of pairs of integers $\mathcal{B}(X) := \{(b_i, r_i)\}$ such that the Riemann-Roch formula may be written as

$$\chi(\mathcal{O}_X(mK_X)) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + l(m),$$

where the correction term $l(m)$ is computed by:

$$l(m) := \sum_{Q_i \in \mathcal{B}(X)} l_{Q_i}(m) := \sum_{Q_i \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{\overline{j}b_i(r_i - \overline{j}b_i)}{2r_i}.$$

Here, we assume that b_i is co-prime to r_i and $0 < b_i \leq \frac{r_i}{2}$. The ratio $\frac{b_i}{r_i}$ is called the *slope* of (b_i, r_i) . For a basket B , we let $\sigma(B) := \sum b_i$ and $\sigma_{12}(B) := \sum_{\frac{b_i}{r_i} \leq \frac{1}{12}} b_i$.

Let

$$\overline{M}^j(b, r) := \frac{\overline{j}b(r - \overline{j}b)}{2r}, \quad M^j(b, r) := \frac{jb(r - jb)}{2r},$$

$$\Delta^j(b, r) := \overline{M}^j(b, r) - M^j(b, r).$$

An easy computation shows that $\Delta^n(b, r) = ibn - \frac{i^2+i}{2}r$, where $i = \lfloor \frac{bn}{r} \rfloor$.

We will need the following easy computational lemmas.

Lemma 4. *Let $b_1r_2 - b_2r_1 = 1$. If $0 < n \neq xr_1 + yr_2$ for any integers $x, y > 0$, then there is no rational number $\frac{b}{n} \in (\frac{b_2}{r_2}, \frac{b_1}{r_1})$ and we have*

$$\Delta^n(b_1 + b_2, r_1 + r_2) = \Delta^n(b_1, r_1) + \Delta^n(b_2, r_2).$$

Proof. We may assume that $\frac{b_2}{r_2} < \frac{b_1}{r_1}$. Note also that by our assumptions, we have $n \leq r_1r_2$. If $\frac{b}{n} \in (\frac{b_2}{r_2}, \frac{b_1}{r_1})$, then $n = (br_2 - b_2n)r_1 + (b_1n - br_1)r_2$ with $br_2 - b_2n > 0$ and $b_1n - br_1 > 0$. Hence, as $n \neq xr_1 + yr_2$ for all $x, y > 0$, then there is no rational number $\frac{b}{n} \in (\frac{b_2}{r_2}, \frac{b_1}{r_1})$.

Let $i_1 := \lfloor \frac{b_1n}{r_1} \rfloor$, $i_2 := \lfloor \frac{b_2n}{r_2} \rfloor$ and $i := \lfloor \frac{(b_1+b_2)n}{r_1+r_2} \rfloor$. If $\frac{b_1n}{r_1}$ is not an integer, then

$$i_2 = \lfloor \frac{b_2n}{r_2} \rfloor \leq i_1 = \lfloor \frac{b_1n}{r_1} \rfloor < \frac{b_1n}{r_1}.$$

If $i_1 \neq i_2$ then $i_1 > \frac{b_2n}{r_2}$ so that $\frac{i_1}{n} \in (\frac{b_2}{r_2}, \frac{b_1}{r_1})$ which is impossible. Therefore $i_1 = i_2 = i$ and the statement follows from the equation $\Delta = ibn - \frac{i^2+i}{2}r$.

If $\frac{b_1 n}{r_1}$ is an integer, then one sees that $\Delta^n(b_1, r_1) = (i_1 - 1)b_1 n - \frac{(i_1 - 1)^2 + (i_1 - 1)}{2} r_1$ so that as $i_2 = i_1 - 1$, the statement follows from the definition of Δ . \square

Lemma 5. *Suppose that $b_1 r_2 - b_2 r_1 = 1$ and suppose that $n = x r_1 + y r_2$ for some integers $r_2 \geq x > 0$, $r_1 \geq y > 0$, then*

$$\Delta^n(b_1 + b_2, r_1 + r_2) = \Delta^n(b_1, r_1) + \Delta^n(b_2, r_2) - \min\{x, y\},$$

Proof. We first remark that the expression $n = x r_1 + y r_2$ for some integers $r_2 \geq x > 0$, $r_1 \geq y > 0$ is unique.

Let $i = x b_1 + y b_2$.

An easy computation shows that if $r_1 \neq y$, then

$$\lfloor \frac{b_1 n}{r_1} \rfloor = \lfloor \frac{x b_1 r_1 + y b_1 r_2}{r_1} \rfloor = \lfloor \frac{x b_1 r_1 + y b_2 r_1 + y}{r_1} \rfloor = i + \lfloor \frac{y}{r_1} \rfloor = i,$$

$$\lfloor \frac{b_2 n}{r_2} \rfloor = \lfloor \frac{x b_2 r_1 + y b_2 r_2}{r_2} \rfloor = \lfloor \frac{x b_1 r_2 - x + y b_2 r_2}{r_2} \rfloor = i + \lfloor \frac{-x}{r_2} \rfloor = i - 1,$$

$$\begin{aligned} \lfloor \frac{(b_1 + b_2)n}{r_1 + r_2} \rfloor &= \lfloor \frac{x b_1 r_1 + y b_1 r_2 + x b_2 r_1 + y b_2 r_2}{r_1 + r_2} \rfloor \\ &= \lfloor \frac{x b_1 r_1 + y b_2 r_1 + y + x b_1 r_2 - x + y b_2 r_2}{r_1 + r_2} \rfloor = i + \lfloor \frac{y - x}{r_1 + r_2} \rfloor. \end{aligned}$$

If $y \geq x$, then $\lfloor \frac{(b_1 + b_2)n}{r_1 + r_2} \rfloor = i$. Direct computation gives

$$\begin{aligned} &\Delta^n(b_1 + b_2, r_1 + r_2) - \Delta^n(b_1, r_1) - \Delta^n(b_2, r_2) \\ &= i(b_1 + b_2)n - \frac{i^2 + i}{2}(r_1 + r_2) - i b_1 n + \frac{i^2 + i}{2} r_1 - (i - 1)b_2 n + \frac{i^2 - i}{2} r_2 \\ &= b_2 n - i r_2 = b_2(x r_1 + y r_2) - (x b_1 + y b_2) r_2 = x(b_2 r_1 - b_1 r_2) = -x. \end{aligned}$$

Note that if $r_1 = y$, then $\lfloor \frac{b_1 n}{r_1} \rfloor = i + 1$. However, one easily sees that the above formula is unchanged.

If $y \leq x$, the computation is similar. \square

Proof of inequality (1). By direct computation, one finds that the K_X^3 and $\chi(\mathcal{O}_X)$ terms coming from the Riemann-Roch formula cancel. Inequality (1) is then equivalent to

$$-3l(2) - l(3) + l(4) + l(5) + l(6) - l(7) \geq 0.$$

Since $l(m) = \sum_{j=1}^{m-1} \overline{M}^j(B) = \sum_{j=1}^m \sum \overline{M}^j(b_i, r_i)$, we must show the inequality

$$\Xi(B) := -2\overline{M}^1(B) + \overline{M}^2(B) + 2\overline{M}^3(B) + \overline{M}^4(B) - \overline{M}^6(B) \geq 0. \quad (3)$$

We will show that this holds for any single basket (b, r) and hence for any basket B .

We define

$$\Xi(B) := -2M^1(B) + M^2(B) + 2M^3(B) + M^4(B) - M^6(B),$$

$$\Xi\Delta(B) := \overline{\Xi}(B) - \Xi(B) = 2\Delta^3(B) + \Delta^4(B) - \Delta^6(B),$$

where we have used the fact that as we assumed that $b/r \leq 1/2$, then $\Delta^1(B) = \Delta^2(B) = 0$.

Step 1. For any single basket $B = \{(b, r)\}$, we have $\Xi(B) = 2b$.

Step 2. For the single basket $B = \{(1, 2)\}$, we have $\Delta^3(1, 2) = 1$, $\Delta^4(1, 2) = 2$ and $\Delta^6(1, 2) = 6$. Hence $\Xi\Delta(B) = -2$, and $\overline{\Xi}(B) = 0$. A similar computation for $B = \{(1, 3)\}$ or $B = \{(1, 4)\}$, yields $\Xi\Delta(B) = -2$ and $\overline{\Xi}(B) = 0$. When $B = \{(1, 5)\}$, then $\Xi\Delta(B) = -1$ and $\overline{\Xi}(B) = 1$.

Step 3. When $B = \{(1, r)\}$ with $r \geq 6$, we have $\overline{M}^m(1, r) = M^m(1, r)$ for all $m \leq 6$. Hence $\overline{\Xi}(B) = \Xi(B) = 2$.

Step 4. Recall that we are assuming $\frac{b}{r} \leq \frac{1}{2}$. Let $S = \{\frac{1}{r}\}_{r \geq 2}$, $S^{(5)} := S \cup \{\frac{2}{5}\}$ and for $n \geq 6$ set

$$S^{(n)} = S^{(n-1)} \cup \left\{ \frac{b}{n} \mid (b, n) = 1, 0 < \frac{b}{n} \leq \frac{1}{2} \right\}.$$

For any $\frac{b}{n} \in S^{(n)}$, let $[0; a_1, \dots, a_t]$ be its continued fraction expression. Note that as $\frac{b}{n} \leq \frac{1}{2}$, then $a_1 \geq 2$. If $t > 1$, we may consider the rational number $\frac{b_1}{r_1}$ with continued fraction expression $[0; a_1, \dots, a_{t-1}]$. We have that $nb_1 - r_1b = \pm 1$ and $\frac{b_1}{r_1} \leq \frac{1}{2}$. Let $b_2 = b - b_1$, $r_2 = n - r_1$. Notice that we also have $b_1r_2 - b_2r_1 = \pm 1$ and $\frac{b_2}{r_2} \leq \frac{1}{2}$. Then we have $\frac{b_1}{r_1}, \frac{b_2}{r_2} \in S^{(n-1)}$.

Step 5. We proceed by showing by induction on r that inequality (3) holds. By Step 1, this is equivalent to showing that $\Xi\Delta(b, r) \geq -2b$.

We have seen that the inequality (3) holds for $r \leq 4$. For $r = 5$, we must consider the single basket $B = \{(2, 5)\}$. Notice that $\Delta^n(2, 5) = \Delta^n(1, 2) + \Delta^n(1, 3)$, for $n = 3, 4, 6$ by Lemma 4. We see that

$$\Xi\Delta(2, 5) = \Xi\Delta(1, 2) + \Xi\Delta(1, 3) = -4.$$

By Step 1, we have $\overline{\Xi}(2, 5) = 0$.

For $r = 6$, there are no new baskets to consider.

Step 6. For $r \geq 7$, notice that by Step 4, we may assume that $(b, r) = (b_1, r_1) + (b_2, r_2)$ for some $r_1, r_2 < r$ and (after possibly switching indices) that $b_1r_2 - b_2r_1 = 1$. By induction hypothesis, we have $\Xi\Delta(b_i, r_i) \geq -2b_i$. Using Lemma 4, it is easy to see that

$$\Delta^m(b, r) = \Delta^m(b_1, r_1) + \Delta^m(b_2, r_2)$$

for $m \in \{3, 4, 6\}$. Hence

$$\Xi\Delta(b, r) = \Xi\Delta(b_1, r_1) + \Xi\Delta(b_2, r_2) \geq -2b.$$

This completes the proof. \square

Proof of inequality (2). The proof is similar but the computations are a little bit more involved.

Inequality (2) is equivalent to

$$-10l(2)-4l(3)+2l(5)+3l(6)-l(7)+l(8)+l(10)-l(11)+l(12)-l(13) \geq 14\sigma_{12},$$

which in turn is equivalent to

$$\begin{aligned} \Xi(B) &:= -9\overline{M}^1(B) + \overline{M}^2(B) + 5\overline{M}^3(B) + 5\overline{M}^4(B) \\ &\quad + 3\overline{M}^5(B) + \overline{M}^7(B) - \overline{M}^{10}(B) - \overline{M}^{12}(B) \geq 14\sigma_{12}(B). \end{aligned} \quad (4)$$

We will show that this holds for any single basket and hence for any basket B .

We define $\Xi(B)$ and $\Xi\Delta(B)$ as in the proof of inequality (1).

Step 1. For any single basket $B = \{(b, r)\}$, we have $\Xi(B) = 14b$.

Step 2. For a single basket $B = \{(1, r)\}$ with $2 \leq r \leq 11$, direct computation gives $\Xi(1, r) = 0, 0, 0, 2, 5, 6, 8, 10, 12, 13$.

Step 3. We claim that if $B = \{(b, r)\}$ with $\frac{b}{r} \leq \frac{1}{12}$, then $\Xi(B) = 14b$.

When $B = \{(1, r)\}$ with $r \geq 12$, we have $\overline{M}^m(1, r) = M^m(1, r)$ for all $m \leq 12$, therefore $\Xi(B) = \Xi(B) = 14$. When $B = \{(b, r)\}$ with $\frac{b}{r} < \frac{1}{12}$ and $b > 1$, as in the proof of inequality (1) Step 4, we may write $b = b_1 + b_2$ and $r = r_1 + r_2$ where b_i and r_i are co-prime, $\frac{b_1}{r_1} > \frac{b}{r} > \frac{b_2}{r_2}$ and $b_1r_2 - b_2r_1 = 1$. By Lemma 4, we see that as $r = r_1 + r_2 > 12$, then $\frac{1}{12} \notin (\frac{b_2}{r_2}, \frac{b_1}{r_1})$. It follows that $\frac{b_1}{r_1} \leq \frac{1}{12}$. The claim now follows by induction. In fact, since $r > 12$, by Lemma 4, we have

$$\Delta^m(b, r) = \Delta^m(b_1, r_1) + \Delta^m(b_2, r_2)$$

for all $1 \leq m \leq 12$.

We proceed by showing by induction on r that $\Xi\Delta(b, r) \geq -14b$. By Step 1, this is equivalent to $\Xi(b, r) \geq 0$ and hence implies that inequality (4) holds.

Step 4. $\Xi(b, r) \geq 0$ for all baskets (b, r) with $r \leq 12$.

By Step 1, $\Xi(b, r) \geq 0$ for all baskets (b, r) with $r \leq 4$. For $r = 5$, we must consider the single basket $B = \{(2, 5)\}$. By Lemmas 4 and 5, one sees that

$$\Delta^n(2, 5) - \Delta^n(1, 2) - \Delta^n(1, 3) = -1, -1, -2, -2 \quad \text{for } n = 5, 7, 10, 12$$

respectively and $\Delta^n(2, 5) - \Delta^n(1, 2) - \Delta^n(1, 3) = 0$ for $n = 1, 2, 3, 4$. It follows that

$$\Xi\Delta(2, 5) = \Xi\Delta(1, 2) + \Xi\Delta(1, 3).$$

By Steps 1 and 2, we have $\Xi(2, 5) = 0$.

For $r = 6$, there are no new baskets to consider.

We can similarly compute all single baskets $B \in S^{(12)} - S^{(6)}$. Recall that each single basket (b, r) , can be compared with pairs (b_1, r_1) and (b_2, r_2) as described in Step 4 of the proof of inequality (1).

We have that $\Xi\Delta(b, r) \geq \Xi\Delta(b_1, r_1) + \Xi\Delta(b_2, r_2)$ for all $B \in S^{(12)} - S^{(6)}$ or more precisely that $\Xi\Delta(b, r) = \Xi\Delta(b_1, r_1) + \Xi\Delta(b_2, r_2) + 1$ if $(b, r) \in \{(3, 10), (5, 12)\}$ and $\Xi\Delta(b, r) = \Xi\Delta(b_1, r_1) + \Xi\Delta(b_2, r_2)$ otherwise. Therefore $\Xi\Delta(b, r) \geq -14b$ for all baskets (b, r) with $r \leq 12$.

Step 5. $\Xi(b, r) \geq 0$ for all baskets (b, r) with $r \geq 13$ and $\frac{b}{r} > \frac{1}{12}$.

We may assume that $(b, r) = (b_1, r_1) + (b_2, r_2)$ for some $r_1, r_2 < r$. By induction hypothesis, we have $\Xi\Delta(b_i, r_i) \geq -14b_i$. By Lemma 4, we have

$$\Delta^m(b, r) = \Delta^m(b_1, r_1) + \Delta^m(b_2, r_2),$$

for $m \leq 12$. Hence

$$\Xi\Delta(b, r) = \Xi\Delta(b_1, r_1) + \Xi\Delta(b_2, r_2) \geq -14b.$$

This completes the proof. \square

We will also need the following equality

Lemma 6. *For any minimal 3-fold of general type with terminal singularities and basket B we have $\sigma(B) = 10\chi(\mathcal{O}_X) + 5P_2(X) - P_3(X)$.*

Proof. The equality follows immediately from the Riemann-Roch formula. \square

3. MAIN RESULT

In this section we prove the main result of this paper.

Theorem 7. *Let X be a smooth 3-fold of general type. Then*

- (1) *There are constants $c' > 0$ and $m_1 > 0$ such that $P_m(X) \geq c'm^3\text{Vol}(X)$ for all $m \geq m_1$.*
- (2) *There is a constant $c > 0$ such that $\text{Vol}(X) \geq c\chi(\mathcal{O}_X)$.*

Proof. We will first prove (1). Consider the Riemann-Roch formula.

If $\chi(\mathcal{O}_X) \leq 0$. Then we get

$$P_m \geq \frac{m(m-1)(2m-1)}{12}\text{Vol}(X) \geq \frac{m^3}{16}\text{Vol}(X)$$

for $m \geq 2$ already.

It remains to consider the case when $\chi(\mathcal{O}_X) > 0$. We will need the following.

Lemma 8. *There exist constants $m_0, c_1, c_2 > 0$ such that*

- (1) $P_{m_0} \geq 2$,
- (2) $P_m \geq 2$ for all $m \geq 5m_0 + 6$,
- (3) $P_m \geq c_1m$ for all $m \geq 12m_0 + 10$ and
- (4) $P_m \geq \frac{c_2m}{t}P_t$ for any $m \geq 10m_0 + 2t + 10$.

Proof. We will repeatedly use the fact that if $P_s > 0$ and $P_t > 0$, then $P_{s+t} \geq P_s + P_t - 1$ and so for all $s \geq t_0 = 5m_0 + 6$ and any $t' > 0$ such that $P_{t'} \geq 2$, we have

$$P_s > \lfloor \frac{s-t_0}{t'} \rfloor (P_{t'} - 1) \geq \frac{s-t_0-t'+1}{t'} (P_{t'} - 1).$$

(1) If $P_i \leq 1$ for $i \in \{5, 6, 8, 10, 12\}$, then by inequality (2), we have $0 < \chi(\mathcal{O}_X) \leq 8$ and $\sigma_{12} = 0$. Since $\sigma = 10\chi(\mathcal{O}_X) + 5P_2 - P_3$ (cf.

Lemma 6), we have $\sigma = \sum b_i \leq 85$. Therefore as $\sigma_{12} = \sum_{\frac{b_i}{r_i} \leq \frac{1}{12}} b_i = 0$, there are only finitely many possible such baskets of singularities and hence there is an integer m_0 such that $P_{m_0}(X) \geq 2$. We may assume that 120 divides m_0 .

(2) If $P_i \geq 2$ for some $i \in \{5, 6, 8, 10, 12\}$, then we have $P_{120}(X) \geq 2$. Therefore, if $\chi(\mathcal{O}_X) > 0$, then $P_{m_0}(X) \geq 2$. By [5] we have that $|mK_X|$ is birational for all $m \geq 5m_0 + 6$.

(3) It follows that for all $m \geq 12m_0 + 10$ we have

$$P_m > \frac{m - 6m_0 - 5}{m_0}(P_{m_0} - 1) \geq \frac{m}{2m_0}.$$

(4) If $P_t = 0$, the proposed inequality is trivial. If $P_t = 1$, the proposed inequality follows from (3) assuming that $c_2 \leq c_1$. We now assume that $P_t \geq 2$ and hence $P_t - 1 \geq P_t/2$. We have that for $m \geq 10m_0 + 2t + 10$,

$$P_m > \frac{m - 5m_0 - t - 5}{t}(P_t - 1) \geq \frac{m}{4t}P_t.$$

□

We have that $2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi(\mathcal{O}_X)$. Hence

$$(2\frac{5}{c_2m} + 3\frac{6}{c_2m} + \frac{8}{c_2m} + \frac{10}{c_2m} + \frac{12}{c_2m})P_m = \frac{58}{c_2m}P_m \geq \chi(\mathcal{O}_X)$$

for any $m \geq 10m_0 + 34$.

Thus, by the Riemann Roch formula

$$(1 + \frac{116}{c_2})P_m \geq P_m + 2m\chi(\mathcal{O}_X) \geq \frac{m^3}{16}\text{Vol}(X).$$

This proves the first inequality.

The second inequality holds trivially if $\chi(\mathcal{O}_X) \leq 0$. Hence we assume that $\chi(\mathcal{O}_X) > 0$. If $P_5, P_6, P_8, P_{10}, P_{12} \leq 1$, then $\chi(\mathcal{O}_X) \leq 8$. Since $| (5m_0 + 6)K_X |$ is birational, then $\text{Vol}(X) \geq \frac{1}{(5m_0+6)^3}$. Therefore,

$$\text{Vol}(X) \geq \frac{1}{(5m_0 + 6)^3} \geq \frac{1}{(5m_0 + 6)^3 \cdot 8} \chi(\mathcal{O}_X).$$

In general, we have $P_{120} \geq P_t$ for $t \in \{5, 6, 8, 10, 12\}$. Hence $8P_{120} \geq \chi(\mathcal{O}_X)$. We may assume that $P_t \geq 2$ for some $t \in \{5, 6, 8, 10, 12\}$ so that $|120K_X|$ is birational. Therefore $120^3\text{Vol}(X) \geq P_{120} - 3 \geq 1$, hence

$$4 \cdot 120^3\text{Vol}(X) \geq 120^3\text{Vol}(X) + 3 \geq P_{120} \geq \frac{1}{8}\chi(\mathcal{O}_X).$$

□

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